THE ABSTRACT ANGLE BETWEEN PYRAMIDAL FINITE ELEMENT SPACES

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Abstract

The finite element multigrid method is the most powerful tool for solving various engineering problems modeled by elliptic partial differential equations. The convergence factor in solving a system of finite element equations by a diagonal block preconditioning depends on the γ-constant in the strengthened Cauchy–Bunyakovsky–Schwarz inequality. The hex-dominant hybrid meshes need transitional elements in the interface subdomain between hexahedral triangulated and simplicial triangulated subdomains. The square pyramids are the most suitable transitional elements. This paper is devoted to refinement strategies for 3D pyramidal finite element meshes and computations of the cosine of the abstract angle between pyramidal finite element spaces.

Keywords: pyramidal finite element spaces, abstract angle between finite dimensional spaces, refinement strategies, congruence classes, refinement tree.

1 INTRODUCTION

The hexahedral finite elements have been preferred to simplicial elements by engineers and practitioners due to their better interpolation properties. But the hexahedral elements are not applicable in the boundary layer because of their larger computational cost. Higher-order simplicial elements are strongly necessary in the boundary layer when curved domains are triangulated. Purely hexagonal meshes essentially increase the complexity of the curved domain modeling problem.

The construction of hex-dominant meshes needs transitional elements between hexahedral and tetrahedral meshes in the interface subdomain. The pyramid and wedge elements are the most popular transition elements since the trace space on the triangular faces of the pyramid and tetrahedra are the same [5]. Additionally, trace space on the quadrilateral faces of the pyramid and hexahedra is identical. These properties of the pyramidal elements make them attractive for the researchers in various scientific areas.

The cosine of the abstract angle between finite element spaces is the convergence factor in multilevel methods based on the diagonal block preconditioning [4]. The contraction number depends only on the geometry of the initial triangulation and the chosen refinement strategy [7,10]. The constant γ in the strengthened Cauchy inequality has been studied by several researchers [1,8,9]. The Brandts et al. [4] conjecture concerning the contraction number for the Laplace operator and red refined Freudenthal's simplicial elements is related to the multidimensional simplicial elements. But the abstract angle between pyramidal finite element spaces has not been calculated up to now unlike the angle between simplicial finite element spaces.

This paper is devoted to the refining of pyramidal meshes and the analysis of the abstract angle between pyramidal finite element spaces. The isotropic diffusion equation is chosen for a model problem. A specific phenomenon generated by the second Sommerville tetrahedron is found.

Further, the paper is organized as follows. Some basic definitions are introduced in Section 2. Refinement strategies related to the pyramidal meshes are described in Section 3. The γ-constant generated by an isotropic diffusion problem is calculated in Section 4. Concluding remarks are discussed in Section 5.

2 PRELIMINARY

Definition 1 *A simply connected domain is said to be canonical if there exists its conforming partition into cubes.*

We denote the second Sommerville tetrahedron [6]

$$\hat{} = \left[\hat{s}_1\left(\frac{\sqrt{2}}{2},0,0\right),\hat{s}_2\left(0,0,\frac{1}{2}\right),\right.$$
$$\left.\hat{s}_3\left(0,0,-\frac{1}{2}\right),\hat{s}_4\left(0,0,-\frac{\sqrt{2}}{2}\right)\right]$$

by $\hat{S}$. The tetrahedron $\hat{S}$ is red invariant [6,11]. The red refinement strategy is illustrated in Figure 1.

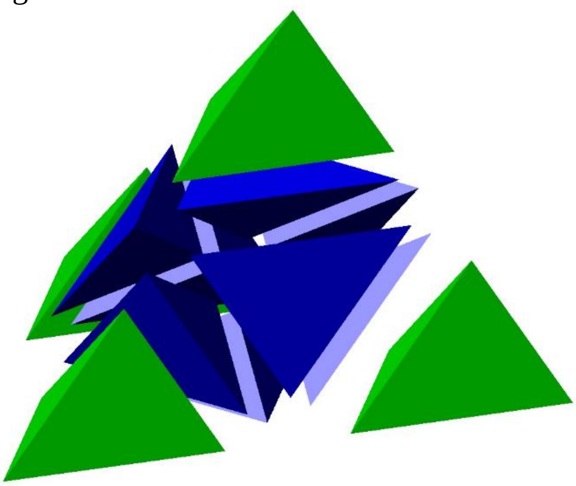

Fig. 1. *The red refinement of a tetrahedron.*

Let the polyhedron *Ω* be conformingly partitioned by the simplicial finite elements T_i, *i=1,2,…m* all of them from the same class *[T]*. Then we write *Ω=mT.*

Definition 2 *Let h(T) be the diameter of the element T, S(T) be the surface area of T and V(T) be the volume of the tetrahedron T. Then the degeneracy measure δ(T) is defined as follows*

$$\delta(T) = \frac{h(T)S(T)}{6V(T)}.$$

The notion measure of degeneracy has been introduced in the three-dimensional case by Zhang [13] in the middle of the nineties. Further, Bey [3] extended this notion in the *n*-dimensional case in the year 2000.

The next definition is related to a completed triangulation.

Definition 3 *The degeneracy measure of a triangulation* τ_k *is defined as follows*

$$\delta(\tau_k) = \max_{T\in\tau_k} \delta(T).$$

Definition 4 *A regular square pyramid is said to be super regular if all edges are the same.*

3 ANALYSES OF 3D PYRAMIDAL MESHES

In this section, we analyze several pyramidal elements. Firstly, we consider the optimal partition of canonical domains.

For this purpose, we divide each cube into six pyramids by the partition operator C, see Figure 2.

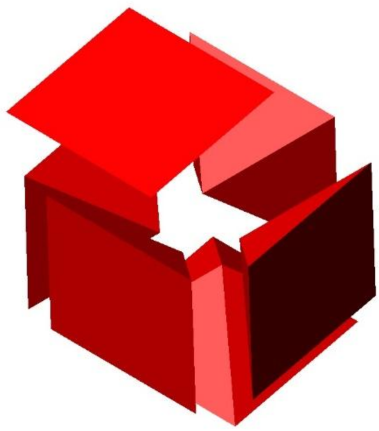

Fig. 2. *The partition of the cube by pyramidal elements.*

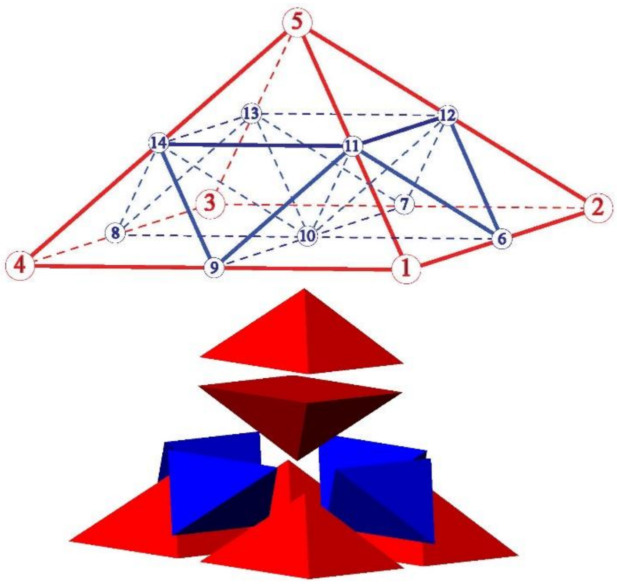

Fig. 3. *The optimal partition of the 3D canonical pyramid.*

Definition 5 *Let*

$$\widehat{K} = [k_1(1,0,0), k_2(1,1,0), k_3(0,1,0), k_4(0,0,0), k_5\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)].$$

The elements of the class $[\widehat{K}]$ *are said to be canonical pyramids.*

The name canonical for the elements of the class $[\widehat{K}]$ arises from the role of these pyramids for triangulating of canonical domains. For each pyramidal finite element P we define the partition operator $\mathcal{L}$ as follows (see Figure 3)

$$\mathcal{L}P = \{[p_1, p_6, p_9, p_{10}, p_{11}], [p_2, p_6, p_7, p_{10}, p_{12}], [p_3, p_7, p_8, p_{10}, p_{13}], [p_4, p_8, p_9, p_{10}, p_{14}], [p_{10}, p_{11}, p_{12}, p_{13}, p_{14}], [p_5, p_{11}, p_{12}, p_{13}, p_{14}], [s_6, s_{10}, s_{11}, s_{12}], \quad [s_7, s_{10}, s_{12}, s_{13}], [s_8, s_{10}, s_{13}, s_{14}], \quad [s_9, s_{10}, s_{11}, s_{14}]\}.$$

If $P \in [\widehat{K}]$ then the first six pyramids belong to the class $[\widehat{K}]$ and the four tetrahedra belong to the class $[\hat{S}]$. Thus, we have

$$\mathcal{L}P = \{6P, 4S | P \in [\widehat{K}], S \in [\hat{S}]\}, \forall P \in [\widehat{K}]$$

with the optimal refinement tree, see Figure 4. Since $\delta(\widehat{K})$=3.41421 and $\delta(\widehat{K})$=2.82843 we obtain

$$\delta(\mathcal{L}C\Omega) = 2 + \sqrt{2}.$$

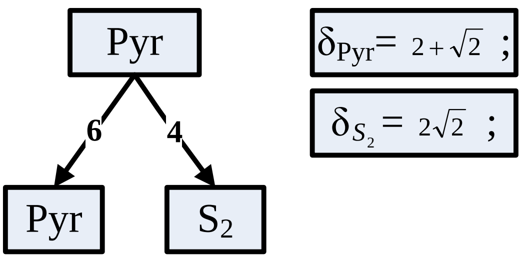

Fig. 4. *The refinement tree generated by the subdivision of the canonical pyramid. The denotations Pyr and S_2 mean correspondingly square pyramid and the second Sommerville tetrahedron.*

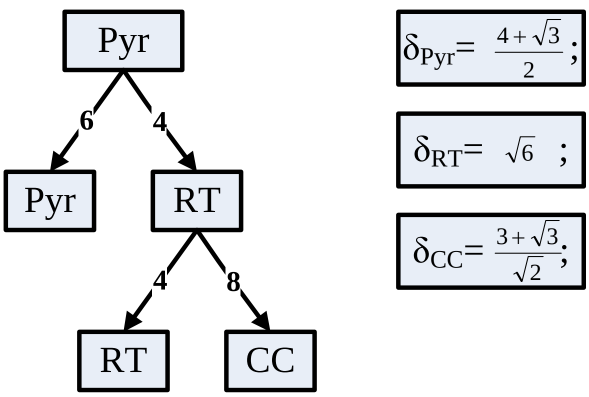

Fig. 5. *The refinement tree generated by the subdivision of the super regular pyramid. Here, Pyr, RT and CC mean correspondingly square pyramid, regular tetrahedron and cube corner.*

We have to compare this result with the corresponding result obtained for the super regular pyramid

$$\hat{R} = [r_1(1,0,0),\ r_2(1,1,0),\ r_3(0,1,0), r_4(0,0,0),\ r_5\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)].$$

By applying the same partition operator on a super regular pyramid R, we obtain the refinement tree presented in Figure 5. In this case, the regular tetrahedra obtained in the first level are refined by the 7-12 refinement strategy created by Todorov in [12]. All regular tetrahedra are red refined and all cube corners are blue refined. We emphasize on the fact that the super regular pyramid generates a more complex refinement tree than the canonical one. The three-level refinement tree in Figure 5 is provided with the following measure of degeneracy

$\delta(\hat{R})$ =2.86603, $\delta(\hat{T})$ =3.34607
and $\delta(\hat{E})$ =2.44949,

where $\hat{T}$ is the canonical cube corner and $\hat{E}$ is a regular tetrahedron. These results are optimal applying pyramidal meshes. Unfortunately, the pyramids from the class $[\hat{R}]$ generates a three-level refinement tree and three classes of similarity.

The pyramid

$$\hat{P} = [\hat{p}_1(-1,-1,0),\ \hat{p}_2(1,-1,0), \hat{p}_3(1,1,0),\ \hat{p}_4(-1,1,0),\ \hat{p}_5(0,0,1)]$$

has been chosen for the reference element in many papers [2,5].

Theorem 1 *The reference element* $\hat{P}$ *belongs to the class* $\widehat{K}$.

Proof. The generic affine transformation of the canonical pyramid is

$$F\ :\ x = A_{\widehat{K}\hat{P}}\hat{x} + B,$$

where

$$A_{\widehat{K}\hat{P}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the transitional matrix of the element $\hat{P}$ is obtained by scaling of an orthogonal matrix. That is why, $\widehat{K}$ and $\hat{P}$ are congruent. □

We do not separately analyse the pyramidal finite element spaces generated by $\hat{P}$ since $\hat{P} \in [\widehat{K}]$.

4 THE ABSTRACT ANGLE BETWEEN PYRAMIDAL FINITE ELEMENT SPACES

Analyzing pyramidal meshes, we essentially use the fact that the γ-constant is the same for all elements of a given class.

Let

$$L(D)u = -\nabla \cdot (D\nabla u)$$

be the elliptic diffusion operator in a canonical domain $\Omega \subset R^3$ with an isotropic diffusion matrix

$$D(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ -\varepsilon & 1 & -\varepsilon \\ 0 & -\varepsilon & 1 \end{pmatrix},\ 0 \le \varepsilon \le \frac{1}{2}.$$

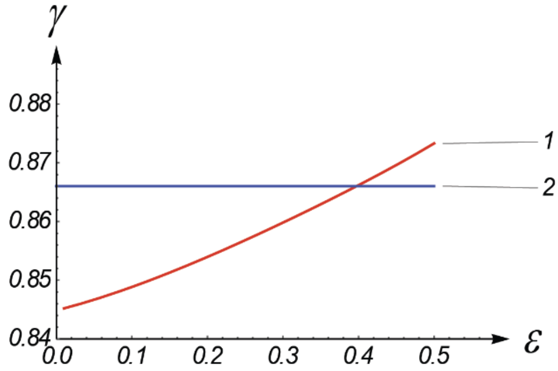

Fig. 6. The graph of the CBS constant obtained by the refinement of the canonical pyramid is denoted by '1'. The corresponding graph for the second Sommerville tetrahedron is denoted by '2'.

The isotropic diffusion problem

$$\begin{cases} L(D)u = f \ \ in\ \ \Omega \\ u = 0 \ \ on\ \ \partial\Omega \end{cases} \qquad (1)$$

with $f \in L^2(\Omega)$ is chosen for a model problem in this section. We write the weak formulation of the problem (1)

$$a(u,v) = (f,v),\ \ \forall v \in H_0^1(\Omega),$$

where $a(u,v)$ is the elliptic bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} d_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx,$$

$(\cdot,\cdot)$ is the L^2-scalar product and $H_0^1(\Omega)$ is the Sobolev space of all functions that are zero on $\partial\Omega$. The coefficients d_{ij} in the bilinear form $a(u,v)$ are the entries of the diffusion matrix.

The contraction numbers obtained by the canonical domain refinement and the bilinear form $a(u,v)$ are presented in Figure 6 and Figure 7.

In Figure 6 we observe a phenomenon created by the second Sommerville tetrahedron. The contraction number $\gamma_S(\varepsilon)$, $S \in [\hat{S}]$ is a constant $\forall \varepsilon \in \left[0, \frac{1}{2}\right]$. This is a unique case valid only for this finite element. Unfortunately, there is no canonical domain, which can only be triangulated by such elements. Sommerville's tetrahedra of the second kind have only supporting roles. The optimal results generated by the pyramidal elements are presented in Figure 7. The same figure indicates that the super regular pyramid has worse contraction properties than the regular tetrahedron.

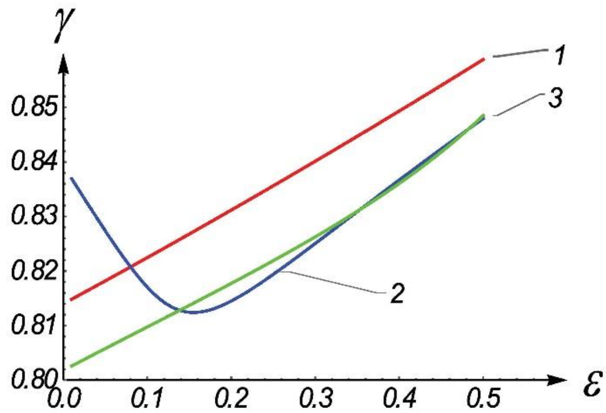

Fig. 7. The graph of the CBS constant obtained by the refinement of the super regular pyramid is denoted by '1'. The corresponding denotations for the cube corner and the regular tetrahedron are '2' and '3'.

5. CONCLUSION

The optimal refinement strategy for the square pyramidal finite element meshes is analyzed. The number of congruence classes and measures of degeneracy are calculated. The isotropic diffusion problem is transformed into a weak form. The contraction number arising from the refinement strategy and the bilinear form in the weak problem is presented graphically. A phenomenon related to the second Sommerville tetrahedron is established. The most useful pyramidal elements generate a simple refinement tree with two congruence classes.

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