International Scientific Conference
UNITECH`2024

A NONLINEAR APPROACH TO TEACHING THE MEAN VALUE THEOREMS
PART I: A DESCRIPTION OF THE METHOD

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Abstract
The Mean Value Theorems (MVTs) are recognized as foundational principles in differential calculus, yet comprehension of their proofs and applications is often challenged by students. Traditional linear teaching methods tend to isolate these theorems, resulting in a fragmented understanding. This study aims to develop a nonlinear teaching approach that highlights the interconnectedness of MVTs and other calculus concepts. Interactive teaching tools and visual aids are employed to enhance students' comprehension by illustrating the relationships between theorems. Significant improvements in engagement and understanding have been indicated by findings, as evidenced by an increased ability of students to apply theorems to real-world problems and to improve their proof-writing skills. Limitations related to diverse learning styles and student populations are acknowledged, while the practical implications of adopting nonlinear methods in calculus education are underscored. This study contributes original insights into pedagogical strategies for teaching MVTs, advocating for a holistic approach that fosters deeper mathematical understanding and retention.

Keywords: mean value theorems, nonlinear teaching approach, calculus education, theorem integration, mathematical understanding.

I. PRELIMINARIES

The Mean Value Theorems (MVTs) are foundational principles in differential calculus, crucial for linking the derivative of a function to its behavior over an interval. Despite their importance, students often face significant challenges in understanding the proofs and applications of these theorems. The traditional linear approach to teaching MVTs, where theorems are introduced sequentially without emphasizing their interconnections, often fails to address these comprehension issues effectively.

Existing solutions to this educational challenge include various teaching strategies that focus on formal proofs, geometric interpretations, and applications in real-world problems [1, 2, 3]. For example, methods such as visual aids, multimedia tools, and interactive examples have been explored to enhance students' grasp of MVTs [4]. However, these approaches retain a linear structure, limiting students' understanding of the broader mathematical context and interrelationships among theorems.

A nonlinear teaching approach, by emphasizing the interconnectedness of MVTs and their relationships with other calculus concepts, shows promise. This method moves beyond rote memorization to foster a deeper understanding of how different theorems complement each other and contribute to a holistic view of calculus.

The main limitation of current teaching practices is their tendency to isolate each theorem, potentially leading to a fragmented understanding of calculus. By focusing on individual proofs and applications, students may struggle to see the overall framework and the integrative nature of these theorems.

This study aims to address this limitation by proposing a nonlinear approach to teaching MVTs. The expected outcome is to enhance students' conceptual understanding and problem-solving skills by illustrating

doi: www.doi.org/10.70456/.........................

how various MVTs interrelate. This approach is anticipated to provide a more cohesive and engaging learning experience, preparing students to tackle complicated calculus problems with a better analytical toolkit.

Duque-Marín et al have highlighted the great difficulty students have experienced in understanding the derivative, a problem compounded when dealing with the MVT [5]. They have proposed that this difficulty may be due to a disconnection between the geometric and analytical interpretations of the MVT. They have used the action-process-object-scheme (APOE) theory to develop two genetic decompositions – graphical and analytical paths – to help overcome these comprehension challenges.

According to Sealey et al examples and visual illustrations can help greatly in understanding these theorems [6]. Insufficient use of such methods in teaching can make the learning process difficult.

Kolahdouz et al have presented students' difficulties in comprehending mathematical proofs, specifically of the Cauchy Generalized Mean Value Theorem [7]. The study is built on research into students' comprehension involving two main aspects: students' understanding of relationships between the statements within the proof of the theorem, and connections with other related theorems.

Dave Ruch has explained the Mean Value Theorem, Roll's theorem, and Cauchy's Mean Value Theorem, including their formal statements, proofs, and geometric interpretations [8]. The author emphasizes the applications of these theorems in understanding the behavior of functions and also presents solutions of practical problems.

J. Li has analyzed the proof and application of Lagrange's Mean Value Theorem within higher mathematics education [9]. The paper highlights the importance of mathematical principles in enhancing both functionality and aesthetics in digital and print media design.

Georges Sarafopoulos has offered an alternative method for teaching MVTs by integrating visual facilities, and interactive examples and focusing on students' conceptual understanding through practical activities and examples [10].

II. INTRODUCTION

Linear (classical) teaching of MVTs is a systematic approach that involves consecutive presenting the theorems in a logical order, ensuring that each concept forms on the previous one.

Our point of view is that nonlinear teaching of MVTs aims to create a better and more interconnected understanding of mathematical concepts. Nonlinear teaching emphasizes deep comprehension over rote memorization and highlights the useful relationship between different theorems.

The basic aims of our research include:

- forming a solid foundation to establish a strong understanding of foundational calculus concepts such as continuity and differentiability;
- enhancing a conceptual understanding to move beyond routine to foster a deeper grasp of theoretical underpinning and applications of the theorems;
- encouraging mastery to prompt mastership of fundamental concepts before moving on to more advanced topics;
- facilitating connections to teach students to make connections between different mathematical concepts and overthink the broader implication of MVTs;
- interactive and engaging learning by using dynamic teaching tools such as graphing software and visualizations to illustrate the geometric aspect of the theorems;
- improving the mathematical rigor and proof skills to ensure students' understanding of formal statements and proofs of MVTs.

III. LINEAR TEACHING APPROACH

This article discusses the classical mean value theorems, which are well-known in the mathematical literature. For the reader's convenience and to introduce additional notation, we will reformulate them here.

Theorem 1. The Weierstrass theorem (WT), (Weierstrass, 1860). Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$ for all $x \in [a, b]$, then $f(x)$ attains both its global minimum and maximum in $[a, b]$, i.e. there exist points μ and ν in $[a, b]$ such that the following statement be true: $f(\mu) \leq f(x) \leq f(\nu)$.

Theorem 2. Rolle's theorem (RT), (Rolle, 1691). Let a function $f(x)$ be:

- continuous on a closed interval $[a, b]$;
- differentiable in each inner points of (a, b);
- $f(a) = f(b)$.

Then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

Theorem 3. Cauchy's mean value theorem (CMVT), (Cauchy, 1821). Let functions $f(x)$ and $g(x)$ be:

- continuous on the closed interval $[a, b]$;
- differentiable on the open interval (a, b);
- $g'(x) \neq 0$ for all x in the open interval (a, b).

Then there exists a point ξ in (a, b) such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (1)$$

Theorem 4. Lagrange's mean value theorem (LMVT), (Lagrange, 1797). Let a function $f(x)$ be:

- continuous on the closed interval $[a, b]$;
- differentiable on the open interval (a, b).

Then there exists a point ξ in (a, b) such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

Theorem 5. The mean value theorem for integrals (MVTI), (the 19th century). Let a function $f(x)$ be continuous on the closed interval $[a, b]$, then there exists a point ξ on the open interval (a, b) such that

$$\int_a^b f(x)\,dx = f(\xi)(b - a).$$

Theorem 6. Cauchy's mean value theorem involving three functions (CMVTITF), (Cauchy, 1823). Let k_1, k_2 and k_3 be three real numbers such that $k_1 + k_2 + k_3 = 0$. If $f(x)$, $g(x)$ and $h(x)$ are three functions satisfying the following conditions:

- continuous on the closed interval $[a, b]$;
- differentiable on the open interval (a, b);
- $f(a) \neq f(b), g(a) \neq g(b)$ and $h(a) \neq h(b)$.

Then there exists a point ξ in (a, b) such that

$$\frac{k_1}{f(b) - f(a)} f'(\xi) + \frac{k_2}{g(b) - g(a)} g'(\xi) + \frac{k_3}{h(b) - h(a)} h'(\xi) = 0 \quad (2)$$

Theorem 7. Cauchy's mean value theorem involving *n* functions (CMVTINF). Let $k_1, k_2, \dots, k_n$ be n real numbers such that $k_1 + k_2 + \dots + k_n = 0$. If $f_1(x), f_2(x), \dots, f_n(x)$ are n functions:

- continuous on the closed interval $[a, b]$;
- differentiable on the open interval (a, b);
- $f_i(a) \neq f_i(b)$ for $i = 1, 2, \dots, n$.

Then there exists a point ξ in (a, b) such that

$$\frac{k_1}{f_1(b) - f_1(a)} f_1'(\xi) + \frac{k_2}{f_2(b) - f_2(a)} f_2'(\xi) + \dots + \frac{k_n}{f_n(b) - f_n(a)} f_n'(\xi) = 0.$$

IV. NONLINEAR TEACHING APPROACH

In our proposed approach, we enhance the traditional teaching of classic theorems by emphasizing the relations between them. This method aims to deepen students' understanding by illustrating how these theorems interrelate and complement each other within the context of calculus. By highlighting these relationships, students grasp the individual significance of each theorem. They also gain insights into how these theorems collectively contribute to a more comprehensive understanding of calculus concepts and applications. This approach fosters a more holistic and interconnected view of calculus, preparing students to deal with complicated problems with a broader analytical toolkit.

Theorem 8. Lagrange's mean value theorem follows from the mean value theorem for integrals.
Proof. We use an integral representation of the function $f(x) = \int_a^x f'(t)dt + f(a)$.
Therefore

$$\int_a^b f'(t)dt = f(t)\Big|_a^b = f(b) - f(a). \quad (3)$$

Applying MVTI to the derivate function $f'(x)$ leads to

$$\int_a^b f'(x)\,dx = f'(\xi)(b-a)$$

It remains to apply (3) in order to complete the proof

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

Thus

$$f(b) - f(a) = f'(\xi)(b-a). \blacksquare$$

This is exactly the statement of LMVT.

Theorem 9. The mean value theorem for integrals follows from Lagrange's mean value theorem.
Proof. We define an auxiliary function

$$F(x) = \int_a^x f(t)dt.$$

The function $F(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), and according to the fundamental theorem of calculus the derivative $F'(x) = f(x)$.

Applying LMVT to $F(x)$ on $[a, b]$, it is stated there exists a point ξ in (a, b) such that

$$F(b) - F(a) = F'(\xi)(b-a).$$

Since

$$F(b) = \int_a^b f(t)dt,$$
$$F(a) = \int_a^a f(t)dt \text{ and}$$
$$F'(\xi) = f(\xi),$$

we get

$$\int_a^b f(t)dt - \int_a^a f(t)dt = f(\xi)(b-a).$$

Thus

$$\int_a^b f(x)dx = f(\xi)(b-a). \blacksquare$$

Theorem 10. Fermat's theorem (FT). Let a function $f(x)$ is defined on the open interval (a, b). If $f(x)$ has a local minimum or a local maximum at a point ξ in (a, b), then the derivate $f'(x) = 0$, or it do not exist.

Connecting WT to CMVT involves understanding the fundamental properties of continuous functions on closed intervals and using these properties to establish the conditions stated in CMVT.
We apply FT to prove the next theorem.

Theorem 11. Cauchy's mean value theorem follows from the Weierstrass theorem

Proof. We define an auxiliary function

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x).$$

Since $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$, and differentiable on (a, b), the function $h(x)$ is also continuous on $[a, b]$, and differentiable on (a, b). According to WT the function $h(x)$ attains a minimum and a maximum in $[a, b]$.

Let x_0 be an extreme point (minimum or maximum). If x_0 is attained at the endpoints a or b, we check if $h(a) = h(b)$.

We calculate $h(a)$ and $h(b)$

$$h(a) = f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} g(a)$$

$$h(b) = f(b) - \frac{f(b) - f(a)}{g(b) - g(a)} g(b).$$

Simplifying $h(a)$ and $h(b)$, we get $h(b) - h(a) = 0$.

If x_0 is an inner point of (a, b), then according to FT, and since the function $h(x)$ has an extremum at a point x_0, we have $h'(x_0) = 0$.

We differentiate $h(x)$ with respect to x

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x).$$

Setting $h'(x_0) = 0$, we have

$$h'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x_0) = 0.$$

Rearranging the latter equality, we get

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Therefore, there exists a point $\xi = x_0$ on the open interval (a, b) such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \blacksquare$$

The relation between WT and RT is classic and fundamental. WT guarantees that a function has global maximum and minimum values on a closed interval, while RT is used to locate a point where the function derivate is equal to zero. Connecting RT to CMVT is also a classic topic in mathematical analysis. CMVT applies to any two functions satisfying the required conditions, while RT is specifically about a single function that attains the same value at the endpoints of an interval.

Theorem 12. Roll's theorem follows from Lagrange's mean value theorem

Proof. If in LMVT, the values of the function $f(x)$ at the endpoints of $[a, b]$ are equal, i.e. $f(a) = f(b)$, then the average rate of change of the function is equal to zero

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a}.$$

We get $f'(\xi) = 0$.

Thus, when $f(a) = f(b)$, according to LMVT, there exists a point ξ in (a, b), such that $f'(\xi) = 0$. ■

V. CONCLUSION

Interactive, visually-driven teaching methods significantly enhance students' conceptual understanding of the Mean Value Theorems. By focusing on these theorems' geometric and real-world applications, students are better equipped to apply them in problem-solving scenarios. Additionally, nonlinear approaches that emphasize connections between different theorems foster greater engagement and improve students' ability to write formal proofs. However, the study acknowledges the need for retention and the effectiveness of these methods across different student populations.

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